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Deriving structural theorems and methods using Tellegen's theorem and combinatorial representations

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Abstract

The paper shows that there are theorems and methods in structural engineering that can be derived from Tellegen's theorem of network graphs. This is demonstrated by deriving from this theorem, Betti's law and the known method for analyzing displacements of truss joints.

This work is a part of a general research approach in accordance with which combinatorial representations (CR) were developed and then applied to represent various engineering systems. In doing so, new connections between engineering fields that traditionally are considered to be unrelated are found. These connections enable augmentation of engineering knowledge in one engineering field by using analogous knowledge from another. This issue is demonstrated in the paper by applying knowledge and methods from electricity to structural mechanics and from machine theory to truss analysis on the basis of the connections between the corresponding CR of these fields.

Keywords: Tellegen's theorem; Graph theory; Combinatorial representations; Structural theorems

1. Introduction

The work reported in the paper is part of a general approach, called – multidisciplinary combinatorial approach, which uses combinatorial representations (CR), for a variety of applications (Shai, 2001b). In the course of this research, CR were developed, the properties of each were examined, and a comprehensive investigation was carried out to establish the connections between them. These representations were then applied to solving engineering problems. This was done by searching among the CR for those which are isomorphic to given engineering problems.

This approach was found to be useful in many aspects, some of which are: checking the validity of engineering systems (Shai and Preiss, 1999a); developing new types of representations in artificial intelligence (Shai and Preiss, 1999b); proving that known algorithms in structures can be derived and even considered to be special cases of known algorithms in CR (Shai, 1999). The CR that are used by the approach include

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Nomenclature

$\mathbf{0}$	zero matrix
A(e)	cut area of rod e
$\vec{\mathbf{A}}$	reduced vector incidence matrix
$\mathbf{B}$	scalar circuit matrix
$\vec{\mathbf{B}}$	vector circuit matrix
CR	combinatorial representations
D(e)	deformation of rod e
$\vec{\mathbf{D}}$	vector of deformations
dim(G)	the dimension of the graph
dr(G)	number of edges with known potential difference values
E(e)	modulus of elasticity of rod e
E(G)	set of edges of graph G
e(G)	number of edges in graph G
$\vec{\mathbf{F}}$	vector of flows
$\vec{F}$(e)	flow in edge e
$\vec{F}_{\mathrm{P}}$	vector of flows in the flow sources
F(e)	value of the flow in edge e
G	graph
G(e)	conductance of edge e
$\mathrm{G_F}$	flow graph
G_Δ	potential graph
$\mathrm{G}_\Delta^\mathrm{R}$	real potential graph
$\mathrm{G_F^V}$	virtual flow graph
$\mathrm{G_R}$	resistance graph
L(e)	length of rod e
MCA	Multidisciplinary combinatorial approach
$\vec{\mathbf{Q}}$	vector scalar matrix
$\mathbf{Q}$	scalar cutset matrix
$\hat{r}$(e)	unit vector in the direction of edge e
R(e)	resistance of edge e
T	statically determinate truss
V(G)	set of the vertices of graph G
v(G)	number of vertices in graph G
$\vec{\boldsymbol{\Delta}}$	vector of potential differences
$\vec{\Delta}$(e)	potential difference in edge e
$\vec{\pi}$(i)	potential of vertex i
$\vec{\boldsymbol{\pi}}$	vector of potentials

graphs, matroids and discrete linear programming. The current paper is entirely dedicated to the graph representations, although the matroid theory is also applicable to structural analysis (Shai, 1999; Kaveh, 1995). Matroid theory enables to obtain a general perspective on problems from graph and network theories (Iri and Tomizawa, 1975; Shai, 2001b).

The first general use of graph theory in engineering appears to be due to Kron (1963), who showed the analogy between electrical networks and elastic structures. He utilized graphs and networks to obtain a uniform method for analysis of large-scale systems. The idea of Kron was to tear the system into several components by means of the corresponding graphs, solve them separately and then to recombine the solutions until the solution of the entire system is obtained. Another unified approach that employs networks was suggested by Wang and Bjorke (1991). The aim of his work was to establish a unified theory to govern a manufacturing system. The network concept was considered as a unified principle in engineering also by Branin (1977).

A computational approach based on graph theory was suggested by Fenves and Branin (1963) and Fenves et al. (1964). On the basis of this approach he developed the software system called "STRESS". Fenves and Gonzalez-Caro (1971) and Munro (1977) employed graph theory in plastic analysis and design. For dynamical systems a "vector-network model" was established by Andrews (1971) on the basis of applying vector mechanics and graph theory to formulation of dynamic motion equations. Kaveh has also employed graph theory (Kaveh, 1991) and was the first one to apply matroid theory for developing efficient structural analysis (Kaveh, 1995, 1997). Kaveh has worked on the improvement of the sparsity of the flexibility matrix by using the correlation between the set of self-equilibrating stress systems and the cycle basis of the graph model of the structure.

There are several unique contributions provided by using the CR, whose comprehensive description can be found in (Shai, 2001a,b). The current paper focuses on application and study of only two of these representations. First of them is the ability to convert methods and theorems from one engineering field to another. This can be done when two domains are represented by the same CR thus enabling the following process: the knowledge from one field is supplied to the common representation, generalized using the inherent properties of the CR and then applied to the other domain represented by the same representation. This ability is employed in this paper when the resistance graph representation (RGR) is applied to represent both electrical circuits (Shai, 2001b) and structures (Shai, 1999). Since electricity domain possesses Tellegen's theorem, it is supplied to the corresponding representation, where it is generalized and then applied to derive the formula for joint displacement (Section 3.4) and Betti's law (Section 3.5). The second advantage of using these CR is the ability to derive new connections between engineering fields on the basis of the connections between their corresponding CR. Such connections have been established when two engineering domains were represented by two mathematically interrelated representations. Consequently, same relations were established between the two represented engineering domains themselves. On the basis of this concept a new connection between mechanisms and determinate trusses was derived since their corresponding CR were found to be dual (Shai, 2001a). This property is employed in the paper where instead of analyzing a determinate truss its dual mechanism was analyzed utilizing methods from machine theory (Section 4.1).

2. Combinatorial representations

The CR that are used in this paper are based on network and graph theories, the relevant details of which can be found in (Shai and Preiss, 1999b) or books on graph theory, such as (Swamy and Thulasiraman, 1981).

Before approaching the CR themselves, a brief review of the definitions will be given. The paper uses terms from network theory where graphs are usually described by matrices such as cutset, circuit and incidence matrices (Balabanian and Bickart, 1969). In the current approach, graphs are used to represent both the topology and the geometry of the engineering system, thus the matrices are resolved in two corresponding types: vector and scalar matrices. The first type is actually the known matrices that are used in the network theory, where the term 'vector' stands for the fact that these matrices provide information about the topological relations between the vectors without considering the geometry of the corresponding elements. The second type of matrices – scalar, provides the information about the geometry of the

Table 1
Graph CR and their usage

	FGR	PGR	RGR
Vertex variables		Potential – $\vec{\pi}$	Potential – $\vec{\pi}$
Edge variables	Flow – $\vec{F}$	Potential difference – $\vec{\Delta}_{\text{head,tail}} = \vec{\pi}_{\text{tail}} - \vec{\pi}_{\text{head}}$	Potential difference – $\vec{\Delta}_{\text{head,tail}} = \vec{\pi}_{\text{tail}} - \vec{\pi}_{\text{head}}$
Flow law – sum of flows in every cutset is equal to zero: $\vec{\mathbf{Q}} \cdot \vec{\mathbf{F}} = \mathbf{0}$	Satisfied	Not satisfied	Satisfied
Potential law – sum of potential differences at every circuit is equal to zero: $\vec{\mathbf{B}} \cdot \vec{\mathbf{\Delta}} = \mathbf{0}$	Not satisfied	Satisfied	Satisfied
Terminal equations			$\vec{\Delta}(\text{e}) = R(\text{e}) \cdot \vec{F}(\text{e}); \vec{F}(\text{e}) = G(\text{e}) \cdot \vec{\Delta}(\text{e})$
Represented engineering systems	Determinate trusses, Static systems	Mechanisms	Indeterminate trusses, dynamic systems, hydraulic systems, integrated systems
Publications in which the representation has been introduced	Shai (2001a,b)	Shai (2001a,b)	Shai (1999, 2001b), Shai and Preiss (1999b)

corresponding engineering elements. These matrices can be obtained from the vector matrices by multiplying each of the non-zero members with a unit vector in the direction of the edge corresponding to the member's column. These definitions were found to be useful since they help to reveal the topological and geometrical properties embedded in the graphs.

Edges in the graphs of the current paper are designated in accordance with their properties as follows: *a solid line* – represents an edge with unknown value of flow or potential difference; *a bold line* – represents an edge for which the flow or potential difference is known; *a dashed line* – represents a chord, which is an edge not included in the spanning tree; *a double line* – represents a branch of a spanning tree.

Table 1 reviews three CR that are used in this paper: Flow graph representation (FGR), potential graph representation (PGR) and the RGR.

2.1. Resistance graph representation for structures

Since the issue of representing a truss by the RGR is fundamental for this paper, the current subsection provides a deeper insight into it.

An important property that is associated with resistance graph is the orthogonality principle that states the following relation between the cutset and circuit matrices of a graph (Swamy and Thulasiraman, 1981):

$$\vec{\mathbf{Q}}^{\text{t}} \cdot \vec{\mathbf{B}} = \mathbf{0} \tag{1}$$

Eqs. (2) and (3) are the immediate outcomes of the orthogonality principle (Shai, 1999) and are used in the current paper.

$$\vec{\Delta} = \vec{\mathbf{Q}}^{\text{t}} \cdot \vec{\Delta}_{\text{T}} \tag{2}$$

$$\vec{\mathbf{F}} = \vec{\mathbf{B}}^{\text{t}} \cdot \vec{\mathbf{F}}_{\text{C}} \tag{3}$$

where $\vec{\Delta}_{\text{T}}$ is the vector of potential differences in the edges of the spanning tree and $\vec{\mathbf{F}}_{\text{C}}$ is the vector of flows in the edges of the chords.

The stages for representing a truss by a resistance graph can be found in (Shai, 1999). The physical meaning of the vertex potential is the vector of displacement of the corresponding joint. The dependence (resistance or conductance) between the flow and the potential difference of the truss edge is based on Hooke's law:

$$D(\mathrm{e}) = \frac{L(\mathrm{e})}{A(\mathrm{e})E(\mathrm{e})}|\vec{F}(\mathrm{e})| \tag{4}$$

where D(e) is the deformation of the rod e, L(e) is the initial length of the rod, A(e) is the section area of the rod, and E(e) is the modulus of elasticity of the rod.

Under the small deflection assumption (West, 1993), the angle change of the rod can be neglected when dealing with axial forces. Nevertheless it cannot be neglected when dealing with the deformation, since the deformation is also relatively small. Hence the potential difference in the rod differs from its length change and there is no explicit dependence between the magnitudes of the flow and the potential difference. Therefore, the terminal equation is to be expressed by means of a matrix.

Let $\Delta_i(\mathrm{e})$ and $F_i(\mathrm{e})$ be the ith coordinate components of the potential difference and the flow at edge e. The general truss member under deformation is shown in Fig. 1.

Under the small deflection assumption, one can derive the following equation connecting the deformation (equal to the magnitude of the potential difference) and the coordinate components of the edge potential differences:

$$D(\mathrm{e}) = |\vec{\Delta}(\mathrm{e})| = \Delta_x(\mathrm{e})\cos\alpha + \Delta_y(\mathrm{e})\sin\alpha \tag{5}$$

Combining Eqs. (4) and (5) gives:

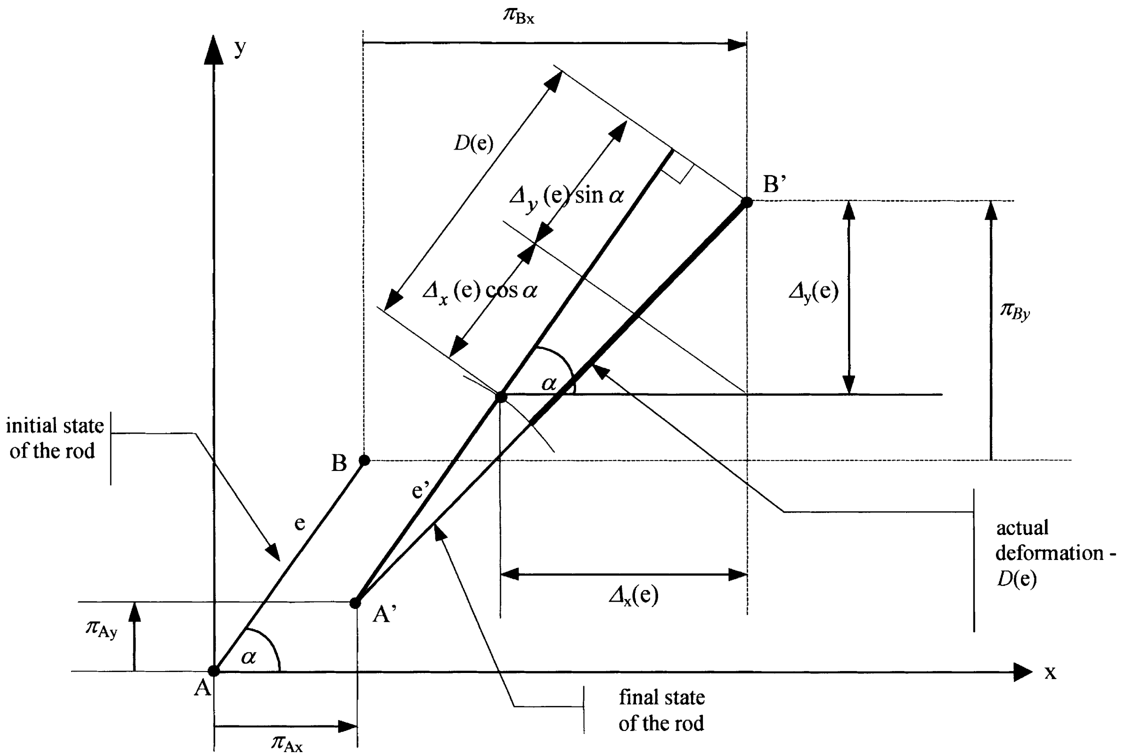


Fig. 1. The description of potential differences of deflected rod.

$$\vec{F}(\mathrm{e}) = \begin{pmatrix} F_x(\mathrm{e}) \\ F_y(\mathrm{e}) \end{pmatrix} = G(\mathrm{e}) \begin{pmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{pmatrix} \begin{pmatrix} \Delta_x(\mathrm{e}) \\ \Delta_y(\mathrm{e}) \end{pmatrix} = G(\mathrm{e})\mathbf{T}(\mathrm{e}) \begin{pmatrix} \Delta_x(\mathrm{e}) \\ \Delta_y(\mathrm{e}) \end{pmatrix} = \mathbf{G}(\mathrm{e}) \begin{pmatrix} \Delta_x(\mathrm{e}) \\ \Delta_y(\mathrm{e}) \end{pmatrix} \tag{6}$$

where

$$G(\mathrm{e}) = \frac{A(\mathrm{e})E(\mathrm{e})}{L(\mathrm{e})}, \quad \mathbf{G}(\mathrm{e}) = G(\mathrm{e})\mathbf{T}(\mathrm{e}) \tag{7}$$

and $\mathbf{T}(\mathrm{e})$ is called 'transformation matrix' of the element.

Thus, the two-dimensional conductance matrix for truss edges becomes a product of the constant conductivity and the transformation matrix. The axial forces in the rods of an indeterminate truss cannot be determined by the laws of statics alone, hence one must also consider the compatibility conditions. In the terminology of combinatorial representation the resistance graph representing the indeterminate truss should be analyzed using both flow and potential laws.

3. Tellegen's theorem

The theorem discussed in this section was developed by Professor BDH. Tellegen (Tellegen, 1952) and therefore bears his name. The main use made of this theorem nowadays is in electric circuit theory (Penfiel et al., 1970; Chua et al., 1987), but it is also used in hydrostatics (Simon et al., 1996), and thermodynamics. In vector-network method (Andrews, 1971) of dynamics, a principle of orthogonality was used, while it was claimed that this principle is an extension of Tellegen's theorem (Andrews and Kesavan, 1978). In electric circuit theory this principle is formulated as follows:

Tellegen's theorem (electrical circuit theory formulation): Let us measure at some time t all voltages $\Delta_k(t)$ and all currents $i_k(t)$ in an electrical circuit, then

$$\sum_k \Delta_k(t) \cdot i_k(t) = \mathbf{\Delta}^{\mathrm{t}}(t) \cdot \mathbf{I}(t) = 0 \tag{8}$$

This paper applies Tellegen's theorem to multidimensional systems and uses it for proving theorems and methods in structural engineering. In order to do that, the theorem is rewritten in the terminology of combinatorial representation, as follows:

Tellegen's theorem (CR formulation): Let $\mathrm{G_F}$ and $\mathrm{G_\Delta}$ be isomorphic flow and potential graphs, then:

$$\sum_{\text{all edges}} \vec{F}^{\mathrm{t}}_{\mathrm{G_F}}(\mathrm{e}) \cdot \vec{\Delta}_{\mathrm{G_\Delta}}(\mathrm{e}) = 0 \tag{9}$$

This theorem is proved in the literature in different ways (Dolan and Aldous, 1993). Two of the proofs are presented in the paper, each giving a different insight on the CR.

3.1. Proof based on the flow law

Starting with the flow law, using the vector incidence matrix gives:

$$\vec{\mathbf{A}}(\mathrm{G_F}) \cdot \vec{\mathbf{F}}(\mathrm{G_F}) = \mathbf{0} \tag{10}$$

The main property of the vector incidence matrix is that in each column there are at most two entries different from zero, namely: '+1' and '−1'. Since a potential is associated with each row, multiplying $\vec{\mathbf{A}}^{\mathrm{t}}$ by the vector of potentials gives the potential difference vector, as follows:

$$\vec{\Delta}^t(G_\Delta) = \vec{\pi}^t(G_\Delta) \cdot \vec{\mathbf{A}}(G_\Delta) \tag{11}$$

therefore:

$$\vec{\Delta}^t(G_\Delta) \cdot \vec{\mathbf{F}}(G_F) = (\vec{\pi}^t(G_\Delta) \cdot \vec{\mathbf{A}}(G_\Delta)) \cdot \vec{\mathbf{F}}(G_F) = \vec{\pi}^t(G_\Delta) \cdot (\vec{\mathbf{A}}(G_\Delta) \cdot \vec{\mathbf{F}}(G_F)) \tag{12}$$

Since G_Δ and G_F are isomorphic, the last expression is equal to zero due to the flow law (Eq. (10)), thus:

$$\vec{\Delta}^t(G_\Delta) \cdot \vec{\mathbf{F}}(G_F) = 0 \tag{13}$$

From this proof, one can see that G_F (G_Δ) should only satisfy the flow (potential) law without restriction on its potentials (flows).

Proof based on the orthogonality principle: On the basis of the orthogonality principle (Eq. (1)) one can perform the following substitution:

$$\vec{\Delta}^t(G_\Delta) \cdot \vec{\mathbf{F}}(G_F) = (\mathbf{Q}^t(G_\Delta) \cdot \vec{\Delta}_T)^t \cdot \vec{\mathbf{F}}(G_F) \tag{14}$$

Applying Eqs. (2) and (3) give:

$$\begin{aligned}\vec{\Delta}^t(G_\Delta) \cdot \vec{\mathbf{F}}(G_F) &= (\vec{\mathbf{Q}}^t(G_\Delta) \cdot \vec{\Delta}_T(G_\Delta))^t \cdot \vec{\mathbf{F}}(G_F) = (\vec{\Delta}_T^t(G_\Delta) \cdot \vec{\mathbf{Q}}(G_\Delta)) \cdot (\vec{\mathbf{B}}^t(G_F) \cdot \vec{\mathbf{F}}_C(G_F)) \\ &= \vec{\Delta}_T^t(G_\Delta) \cdot (\vec{\mathbf{Q}}(G_\Delta) \cdot \vec{\mathbf{B}}^t(G_F)) \cdot \vec{\mathbf{F}}_C(G_F)\end{aligned} \tag{15}$$

since G_Δ and G_F are isomorphic, the last expression is equal to zero due to the orthogonality principle.

3.2. Tellegen's theorem in one-dimensional trusses

Before dealing with multidimensional systems let us consider the interpretation of Tellegen's theorem in a one-dimensional engineering system, because of the simplicity of its explanation.

Example. Fig. 2 shows a one-dimensional statically indeterminate truss, the one-dimensional flow graph representing it and the corresponding matrices.

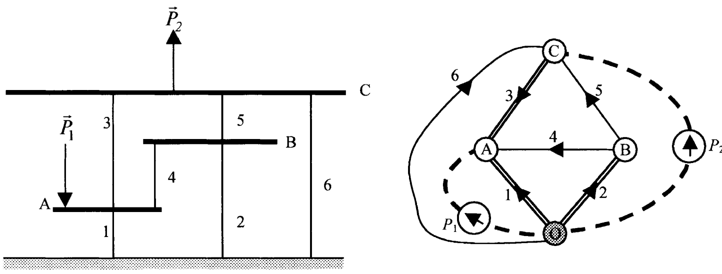
The scalar matrices are obtained from the vector matrices by multiplying each column by its corresponding unit vector. Since the example in Fig. 2 is one-dimensional all the columns are multiplied by either sin(90°) or sin(270°) depending on whether the edge is directed upwards or downwards. One can see that such a multiplication preserves the validity of the orthogonality for the scalar matrices.

Any vector **F** that satisfies the flow law is a feasible flow vector of the graph, hence is a feasible force vector of the truss. The orthogonality principle states: $\mathbf{Q}(G) \cdot \mathbf{B}^t(G) = \mathbf{0}$. Thus every row i of $\mathbf{B}(G)$ satisfies $\mathbf{Q}(G) \cdot \mathbf{B}^t(G)_{i\rightarrow} = \mathbf{0}$, hence it is a feasible vector of forces in the truss, i.e. it is a state of self stress. For example, first row of $\mathbf{B}(G)$ gives the set of forces presented in Fig. 3.

From equation $\mathbf{F} = \mathbf{B}^t(G) \cdot \mathbf{F}_C$, and the orthogonality principle it follows that every feasible force can be obtained from $\mathbf{B}(G)$ as a linear combination of its rows.

In a similar way the rows of $\mathbf{Q}(G)$ are proved to be a set of feasible displacement vectors. Let $\boldsymbol{\Delta}$ be a set of feasible displacements, i.e. it satisfies $\mathbf{B}(G) \cdot \boldsymbol{\Delta} = 0$. On the basis of equation $\boldsymbol{\Delta} = \mathbf{Q}^t(G) \cdot \boldsymbol{\Delta}_T$, and the orthogonality principle, we conclude that every feasible displacement vector is a linear combination of the scalar cutset matrix rows. For example, the first row of **Q** gives the following feasible set of potential differences, as shown in Fig. 4.

On this basis of the above properties of the matrices it follows that the rows of the cutset (circuit) matrix span the linear space of feasible displacements (forces) of the structure. This is derived from the properties



(a) (b)

$$\vec{\mathbf{Q}} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & -1 \end{pmatrix} \quad (\text{columns: } 1\ 2\ 3\ 4\ 5\ 6\ P_1\ P_2)$$

(c)

$$\vec{\mathbf{B}} = \begin{array}{c} 4 \\ 5 \\ 6 \\ P_1 \\ P_2 \end{array}\begin{pmatrix} -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{columns: } 1\ 2\ 3\ 4\ 5\ 6\ P_1\ P_2)$$

(d)

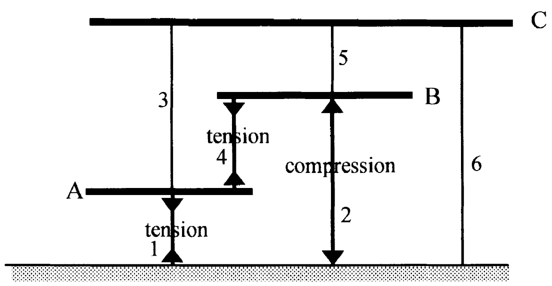
$$\mathbf{Q} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 & -1 \end{pmatrix} \quad (\text{columns: } 1\ 2\ 3\ 4\ 5\ 6\ P_1\ P_2)$$

(e)

$$\mathbf{B} = \begin{array}{c} 4 \\ 5 \\ 6 \\ P_1 \\ P_2 \end{array}\begin{pmatrix} -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{columns: } 1\ 2\ 3\ 4\ 5\ 6\ P_1\ P_2)$$

(f)

Fig. 2. Example of a one-dimensional truss, its graph and the corresponding matrices. (a) One-dimensional truss, (b) corresponding graph, (c), (e) vector and scalar cutset matrices and (d), (f) vector and scalar circuit matrices.



$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_{P_1} \\ F_{P_2} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(a) (b)

Fig. 3. Example of a self stress state corresponding to a row of the scalar circuit matrix. (a) The self stress state, (b) the first row in the scalar circuit matrix.

of these matrices. The results obtained in the following sections are also based on the connections between the two matrices.

3.3. Tellegen's theorem for multidimensional systems

As can be seen from the vectorial form of Tellegen's theorem, the theorem can also be applied to multidimensional flows and potentials, each having a component for each coordinate. The vector cutset (circuit) matrices depend only on the topology of the graph, hence they do not differ from the one-dimensional case. Therefore for two isomorphic multidimensional graphs G_F and G_Δ, the Tellegen's theorem is valid: $\vec{\mathbf{F}}^t(G_F) \cdot \vec{\Delta}(G_\Delta) = 0$, because the orthogonality principle (Eq. (1)) remains unchanged. For the simplicity of the explanations, the paper deals with two-dimensional structures, although the approach is applicable to three-dimensional trusses as well.

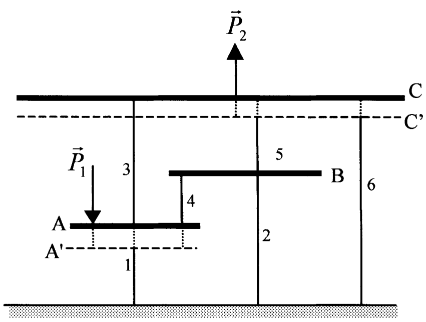
3.3.1. Applying Tellegen's theorem to the resistance graphs representing plane trusses

This section presents a special case of two-dimensional graphs, which represent trusses. In this case, the multidimensional Tellegen's theorem is formulated in scalar form. Eq. (16) is written for each of the dimensions of the resistance graph as follows:

$$\begin{aligned}
\mathbf{F}_x(G_F) \cdot \mathbf{\Delta}_x(G_\Delta) &= \sum_{\text{rods of the truss}} F_{x_i}(G_F)\Delta_{x_i}(G_\Delta) + \sum_{\text{external forces}} P_{x_i}(G_F)\Delta_{P_{x_i}}(G_\Delta) \\
&\quad + \sum_{\text{external reactions}} R_{x_i}(G_F)\Delta_{R_{x_i}}(G_\Delta) = 0 \\
\mathbf{F}_y(G_F) \cdot \mathbf{\Delta}_y(G_\Delta) &= \sum_{\text{rods of the truss}} F_{y_i}(G_F)\Delta_{y_i}(G_\Delta) + \sum_{\text{external forces}} P_{y_i}(G_F)\Delta_{P_{y_i}}(G_\Delta) \\
&\quad + \sum_{\text{external reactions}} R_{y_i}(G_F)\Delta_{R_{y_i}}(G_\Delta) = 0
\end{aligned} \tag{16}$$

For each type of reaction, mobile or fixed support, in each coordinate, one of the multipliers (reaction force or the potential difference) is equal to zero. Therefore, the multiplication is always equal to zero, hence the terms concerned with reactions vanish, and the following equations remain:

$$\begin{aligned}
\mathbf{F}_x(G_F) \cdot \mathbf{\Delta}_x(G_\Delta) &= \sum_{\text{rods of the truss}} |\vec{F}_i(G_F)| \cos\alpha_i \cdot \Delta_{x_i}(G_\Delta) + \sum_{\text{external forces}} |\vec{P}_i(G_F)| \cos\alpha_i \cdot \Delta_{P_{x_i}}(G_\Delta) = 0 \\
\mathbf{F}_y(G_F) \cdot \mathbf{\Delta}_y(G_\Delta) &= \sum_{\text{rods of the truss}} |\vec{F}_i(G_F)| \sin\alpha_i \cdot \Delta_{y_i}(G_\Delta) + \sum_{\text{external forces}} |\vec{P}_i(G_F)| \sin\alpha_i \cdot \Delta_{P_{y_i}}(G_\Delta) = 0
\end{aligned} \tag{17}$$



$$\begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \\ \Delta_{P_1} \\ \Delta_{P_2} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

(a) (b)

Fig. 4. Example of a feasible set of displacements corresponding to a row of cutset matrix. (a) Displacements state, (b) the first row in the scalar cutset matrix.

The potential differences in the edges corresponding to the external forces are equal to the difference between potentials of the reference vertex and of the vertex upon which the external force is applied. Since the potential of the reference vertex is equal to zero, the potential difference of the external force edge becomes equal to the potential of the vertex upon which the force is applied with the minus sign:

$$\vec{\Delta}_{P_i}(\mathrm{G}) = -\vec{\pi}_{P_i}(\mathrm{G}) \tag{18}$$

Summing up Eq. (16) and substituting Eq. (18) gives:

$$\sum_{\text{rods of the truss}} F_i(\mathrm{G_F})\left(\cos\alpha_i\,\Delta_{x_i}(\mathrm{G_\Delta}) + \sin\alpha_i\,\Delta_{y_i}(\mathrm{G_\Delta})\right) - \sum_{\text{external forces}} \vec{P}_i(\mathrm{G_F})\cdot\vec{\pi}_{P_i}(\mathrm{G_\Delta}) = 0 \tag{19}$$

Substituting Eq. (5) to Eq. (19) gives:

$$\sum_{\text{rods of the truss}} F_i(\mathrm{G_F})\cdot D_i(\mathrm{G_\Delta}) - \sum_{\text{external forces}} \vec{P}_i(\mathrm{G_F})\cdot\vec{\pi}_{P_i}(\mathrm{G_\Delta}) = 0 \tag{20}$$

By the definition, the force and the displacement in each truss element are parallel, hence the multiplication of magnitudes in the last equation can be replaced by scalar multiplication of the corresponding vectors:

$$\sum_{\text{rods of the truss}} \vec{F}_i(\mathrm{G_F})\cdot\vec{D}_i(\mathrm{G_\Delta}) - \sum_{\text{external forces}} \overrightarrow{P_i}(\mathrm{G_F})\cdot\vec{D}_{P_i}(\mathrm{G_\Delta}) = 0 \tag{21}$$

The last equation will be referred in the paper as the: "Multidimensional Tellegen's theorem for Trusses".

3.4. Deriving the method for determination of joint displacements from Tellegen's theorem

In this section, it will be shown that the known equation for determining the displacement of a joint in a truss is a special case of the Multidimensional Tellegen's theorem for Trusses.

In order to apply Tellegen's theorem, two of the combinatorial representations that were given in Table 1 will be used. The steps for building the CR are the same as was explained in (Shai, 2001b). An extra edge, called "control edge", is added to both. Its head vertex is the vertex whose displacement is to be determined, and the tail vertex is the reference vertex.

The CR – the flow and potential graphs are used in two different ways as follows:

For the real potential graph $\mathrm{G}_\Delta^\mathrm{R}$ the flow values in the source edges are the values of the external forces. In the control edge we put a 'potential difference measurement', which corresponds to a potential difference measuring device (like voltmeter in electrical circuit) that is located between the end vertices of the corresponding edge. The 'R' superscript over G indicates, that the potential differences in it are due to the "real" external forces applied to the structure.

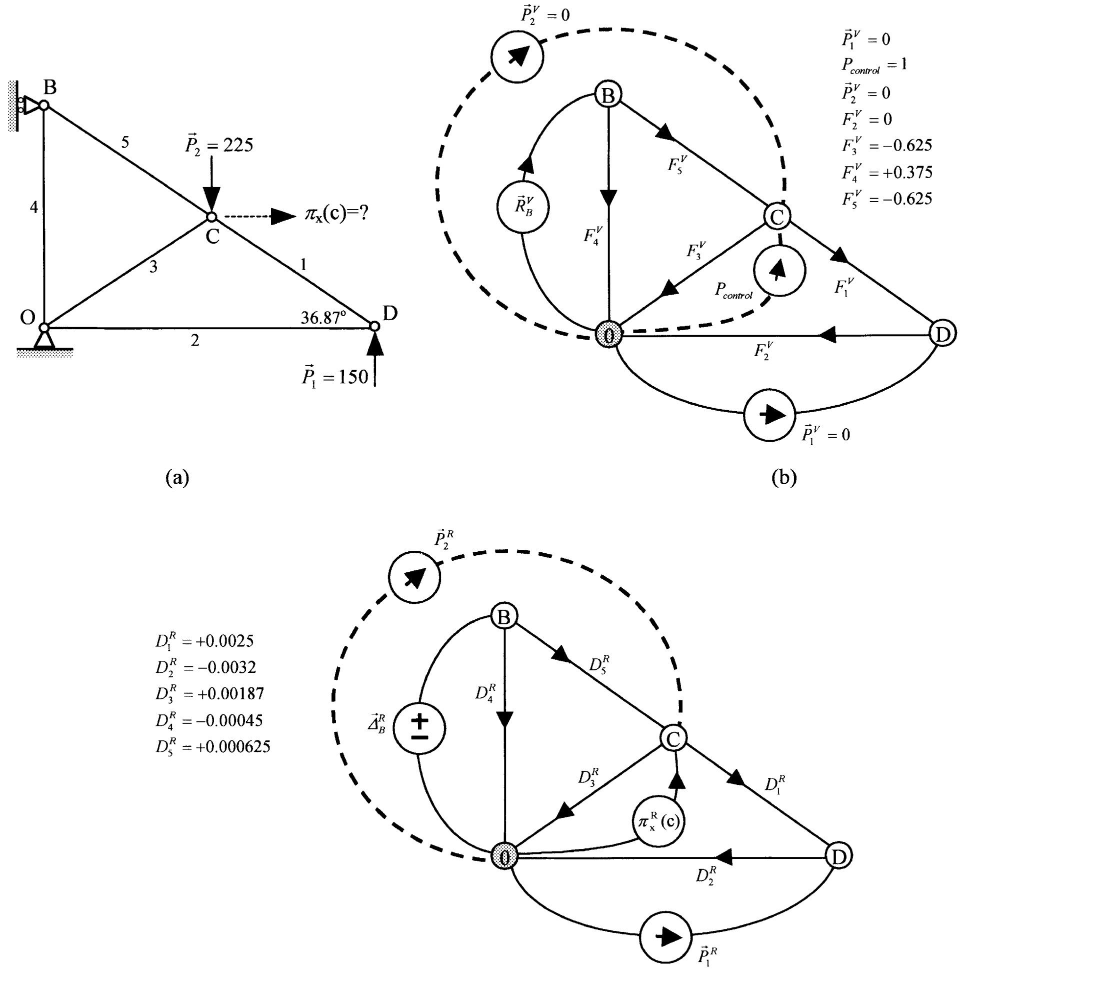
For the virtual flow graph $\mathrm{G}_\mathrm{F}^\mathrm{V}$ all the source edges, which correspond to the external forces are assigned flow sources with values equal to zero. One can think about it as a disconnection. In the control edge, a unit force is applied in the direction of the displacement that has to be measured. The 'V' superscript over G indicates that the flows in the graph are not the real forces in the structure, but the forces due to an arbitrary virtual external force applied onto the structure.

Applying the multidimensional Tellegen's theorem to the two graphs, gives:

$$\sum_{\text{rods of the truss}} \vec{F}_i(\mathrm{G_F^V})\cdot\vec{D}_i(\mathrm{G_\Delta^R}) - \sum_{\text{external forces}} 0\cdot\vec{D}_{P_i}(\mathrm{G_\Delta^R}) - 1\cdot D_{\text{control}}(\mathrm{G_\Delta^R}) = 0 \tag{22}$$

From here, the well-known equation (West, 1993) for analyzing the displacement of a joint is derived:

$$D_{\text{control}}(\mathrm{G_\Delta^R}) = \sum_{\text{rods of the truss}} \frac{F_i(\mathrm{G_F^V})\cdot F_i(\mathrm{G_\Delta^R})\cdot L_i}{A_i\cdot E_i} \tag{23}$$



(c)

$$\pi_x(c) = F_{b0}(G_F^V)\cdot D_{b0}(G_\Delta^R) + F_{bc}(G_F^V)\cdot D_{bc}(G_\Delta^R) + F_{c0}(G_F^V)\cdot D_{c0}(G_\Delta^R) = (0.375\cdot -0.00045) + (-0.625\cdot 0.000625) + (-0.625\cdot 0.001875) = -0.001731[m]$$

(d)

Fig. 5. Example of analyzing joint displacement by the multidimensional Tellegen's theorem. (a) The truss, (b) the virtual flow graph G_F^V, (c) the real potential graph G_Δ^R and (d) calculation of the displacement of joint c in the direction of the x-axis.

An example for applying Eq. (23), is given in Fig. 5, where the horizontal displacement of joint c is to be determined.

3.5. Deriving Betti's law from Tellegen's theorem

The conventional proof of Betti's law is based on energy considerations (West, 1993), while in this section, it is proved on the basis of Tellegen's theorem.

Suppose, a truss and two different sets of external loads applied on it are given. The first set of external loads $\vec{\mathbf{P}}_1$ causes joint displacements, $\vec{\pi}_1$, internal forces $\vec{\mathbf{F}}_1$ and deformations $\vec{\mathbf{D}}_1$. The second set of external loads $\vec{\mathbf{P}}_2$ causes joint displacements $\vec{\pi}_2$, internal forces $\vec{\mathbf{F}}_2$ and deformations $\vec{\mathbf{D}}_2$.

Since both sets of loads act on the same truss, and the forces (potential differences) satisfy the flow (potential) law, then according to Tellegen's theorem (Eq. (9)) multiplication of forces from one set by the potential differences from the other set is equal to zero, as follows:

$$\begin{pmatrix} \vec{\mathbf{F}}_1^{\mathrm{t}} & \vec{\mathbf{P}}_1^{\mathrm{t}} \end{pmatrix} \cdot \begin{pmatrix} \vec{\mathbf{D}}_2 \\ -\vec{\pi}_{P_2} \end{pmatrix} = \vec{0} \rightarrow \vec{\mathbf{F}}_1^{\mathrm{t}} \cdot \vec{\mathbf{D}}_2 = \vec{\mathbf{P}}_1^{\mathrm{t}} \cdot \vec{\pi}_{P_2} \tag{24}$$

$$\vec{\mathbf{P}}_1^{\mathrm{t}} \cdot \vec{\pi}_{P_2} = \vec{\mathbf{F}}_1^{\mathrm{t}} \cdot \vec{\mathbf{D}}_2 \overset{\substack{\text{resistance} \\ \text{relation}}}{=} \vec{\mathbf{F}}_1^{\mathrm{t}} \cdot (\mathbf{R} \cdot \vec{\mathbf{F}}_2) = (\vec{\mathbf{F}}_1^{\mathrm{t}} \cdot \mathbf{R}) \cdot \vec{\mathbf{F}}_2 \overset{\substack{\text{Since R is} \\ \text{diagonal}}}{=} \vec{\mathbf{D}}_1^{\mathrm{t}} \cdot \vec{\mathbf{F}}_2 \tag{25}$$

Another form of the Tellegen's theorem for the two graphs is:

$$\begin{pmatrix} \vec{\mathbf{F}}_2^{\mathrm{t}} & \vec{\mathbf{P}}_2^{\mathrm{t}} \end{pmatrix} \cdot \begin{pmatrix} \vec{\mathbf{D}}_1 \\ -\vec{\pi}_{P_1} \end{pmatrix} = \mathbf{0} \rightarrow \vec{\mathbf{F}}_2^{\mathrm{t}} \cdot \vec{\mathbf{D}}_1 = \vec{\mathbf{P}}_2^{\mathrm{t}} \cdot \vec{\pi}_{P_1} \tag{26}$$

Combining the last two equations gives:

$$\vec{\mathbf{P}}_1^{\mathrm{t}} \cdot \vec{\pi}_{P_2} = \vec{\mathbf{P}}_2^{\mathrm{t}} \cdot \vec{\pi}_{P_1} \tag{27}$$

Eq. (27) is known in the literature as the reciprocity theorem or Betti's law (West, 1993).

4. Using augmented knowledge from combinatorial representations

As mentioned in the introduction, using CR in engineering enables to use a single representation for several different engineering fields. This enables usage of knowledge and methods from one field in the other, as is demonstrated in the current section.

Based on the properties of the FGR (Shai, 2001a), its dual graph corresponds to the PGR. Moreover, it has been proved (Shai, 2001a) that instead of analyzing a determinate truss, one can find its dual mechanism, and perform a velocity analysis on it by using methods and theorems from machine theory. The methodology for building a mechanism dual to a truss is related in a certain way with the process of building the famous Maxwell–Cremona diagrams for a truss (Timoshenko and Young, 1965), but the purpose and the outcome of the two are quite different. The idea of employing these results in truss analysis is carried out in the following subsection.

4.1. Analyzing the joint displacement using the dual mechanism

This issue is explained using the example of Fig. 6. The external force *P* is applied at joint c and the displacement of joint c along the direction of *P* is to be determined.

In order to solve this problem, one should use Eq. (23) developed in Section 3.4. To apply this equation the potential differences in $\mathrm{G}_{\Delta}^{\mathrm{R}}$ and flows in $\mathrm{G}_{\mathrm{F}}^{\mathrm{V}}$ should be determined. The first step is to obtain the deformations of the rods by one of the various methods for analysis of statically indeterminate trusses. This way the potential differences in $\mathrm{G}_{\Delta}^{\mathrm{R}}$ are calculated. The second step is to find any solution that satisfies only the force equilibrium in the truss, when a unit external force is applied at the same location and direction as *P*. This way the flows in $\mathrm{G}_{\mathrm{F}}^{\mathrm{V}}$ are found. The forces can be obtained by setting all the forces in the redundant rods to zero and solving the resultant determinate truss. This procedure significantly simplifies the solutions, since instead of analyzing an indeterminate truss, a determinate truss is analyzed. This leaves us with analysis of a determinate truss, for instance, the one shown in Fig. 7.

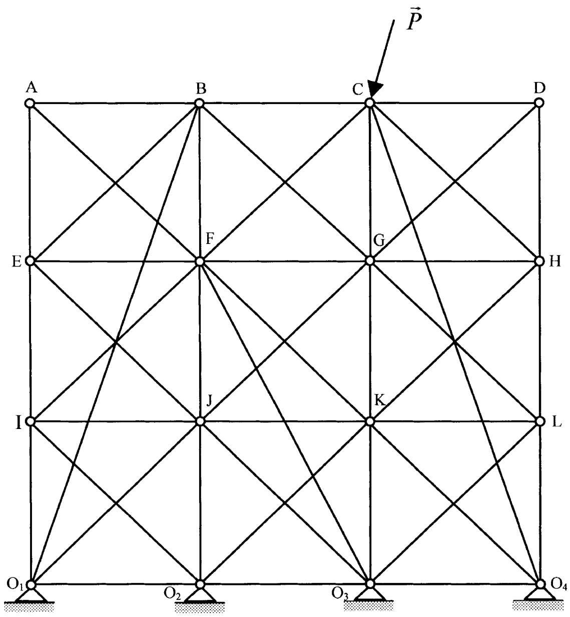


Fig. 6. Statically indeterminate truss.

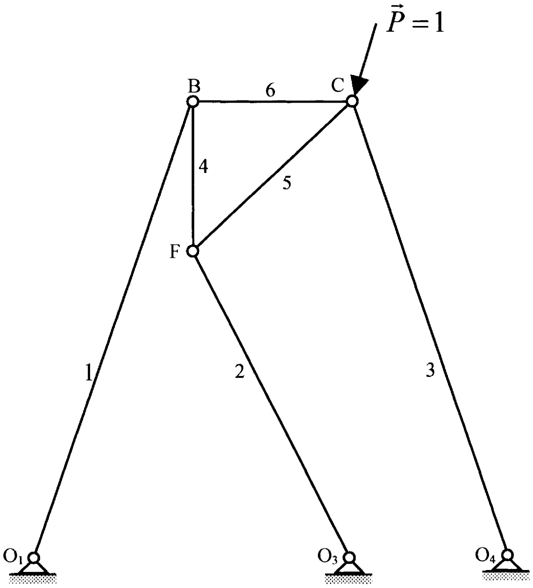


Fig. 7. Determinate truss.

The graphs corresponding to the application of the Tellegens' theorem are shown in Fig. 8.

Using the above-mentioned result obtained in (Shai, 2001a) the analysis of determinate truss can be transformed into the velocity analysis of its dual mechanism shown in Fig. 9.

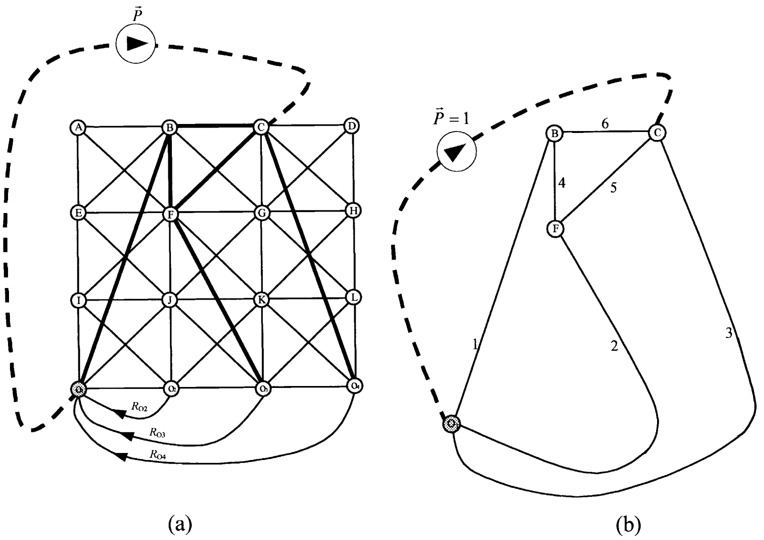


Fig. 8. The graphs corresponding to the truss in Fig. 7. (a) The real potential graph – G_{Δ}^{R}, (b) the virtual flow graph – G_{F}^{V}.

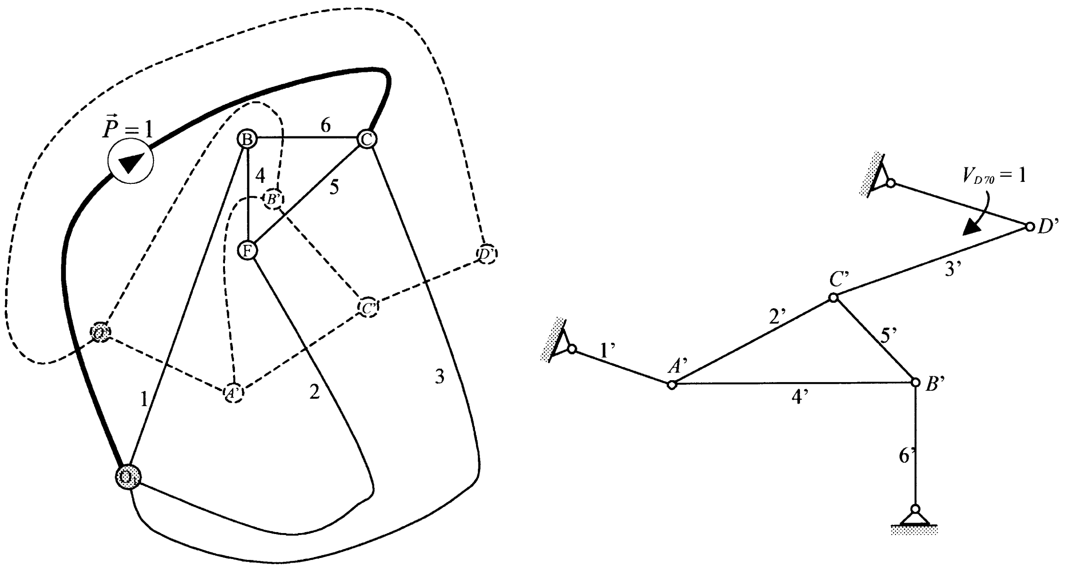


Fig. 9. The mechanism dual to the truss in Fig. 7. (a) The flow graph representing the truss and its dual potential graph superimposed and (b) the dual mechanism.

The solution of the dual mechanism is immediate, since it is the known mechanism called "Stephenson type III" (Erdman and Sandor, 1997), for which there is an efficient solution method, where the driving link is exchanged and the image velocity diagram is drawn, as shown in Fig. 10.

The forces in the initial truss are proportional to the corresponding lengths in the image velocity diagram of Fig. 10 as follows:

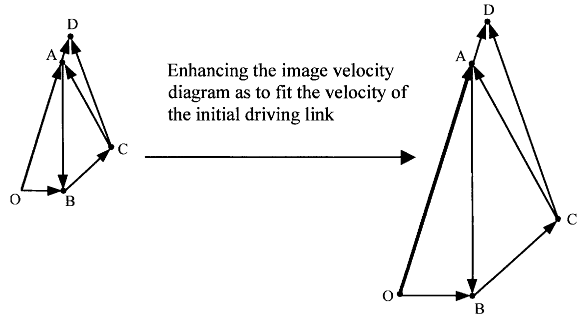


Fig. 10. Building the image velocity diagram for the mechanism shown in Fig. 9.

$$F(\mathrm{e}) = \mu \cdot \mathrm{e}' \tag{28}$$

where e′ – the length corresponding to link e measured from diagram in Fig. 10 and $\mu = (P/\mathrm{od}) = (1/\mathrm{od})$

5. Conclusions

The paper has shown that when structural systems are represented by CR, theorems inherent in these representations can be used to prove properties of the structural systems. The paper used two CR: the flow and resistance graphs. Using Tellegen's theorem from network theory, and other properties of CR, a known method for determining the displacements of the truss joints was derived in a different way. Also, an alternative method of proving theorems of structural mechanics was given in the paper by showing that Betti's law is derived from Tellegen's theorem.

Using CR enables one to obtain a global perspective of different engineering fields, by finding novel connections between them on the basis of the connections between their corresponding CR. In the example shown in the paper, the trusses are represented by the FGR and mechanisms by the PGR. Since these representations have been proved to be dual, it follows that trusses are dual to mechanisms. Accordingly, the analysis process for the truss was transformed to the velocity analysis of its corresponding dual mechanism, which enabled to employ methods and algorithms developed in machine theory.

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